A Game Theoretic Approach to Student Effort Allocation Across Courses

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Abstract

In this paper, I try to address how students would react to the simultaneous presence of different grading schemes. I take two types of grading schemes, absolute grading, which rewards based on a threshold effort value, and relative grading, which rewards based on the relationship a student's effort has with respect to the competing student. To achieve this, I incorporate the Colonel Blotto game for measuring the best payoffs students can earn from relative grading classes. Specifically, I treat it as a sub-game within a game that students can use to derive their expected payoffs from to judge how they should split effort between the two types of classes should they be taking them simultaneously. Through this, I address the basic dynamics of student effort allocation across different courses with different grading schemes. In addition, I consider a situation where the students have different thresholds of reward for the absolute grading class. I find that there exists an equilibrium where the student who has the higher threshold for the absolute grading class is better off should the threshold be high enough and the payoffs of the classes fall under a specific relation.

Introduction

The question of what the optimal allocation of effort should be against an opponent on multiple fields of competition has probably been extensively explored by policy makers since the first competition between two human civilizations; most likely in the form of "Should we invest labor in our troops, harvest, or cultural activities?".

On the other hand, the first mathematical formalization of the question, presently known as the Colonel Blotto game, had not been achieved until the mid 20th century in Gross and Wagner (1950). The paper discusses the optimal troop distribution strategy for two army commanders or players (one of which is Colonel Blotto) who are looking to win over as many battlefields as possible in a zero sum game by deploying more troops than the enemy commander to each battlefield. In particular, the paper considers a two battlefields case with symmetric and asymmetric troops reserves and three or more battlefields case with symmetric troop reserves.

Due to the nature of exponentially increasing number of strategies with respect to the number of battlefields that become available, solving for more general n battlefield games, accommodating for asymmetry of reserve troops, appeared after decades in Roberson (2006), which utilizes n-copula distributions.

Nonetheless, the depth of insight one can bring even with results regarding two battlefields with mathematical reasoning is impressive. In this paper, I aim to apply the results of analyzing a two person competition across two battlefields from Gross and Wagner (1950) to a hypothetical situation of competition between two students across several courses that vary in grading scheme between absolute grading and relative grading.

I would primarily like to see how two competitors competing over multiple battlefields would react to a mix of grading schemes. Specifically, I will gradually develop a game from a simple form of single type classes where there are only absolute grading classes or only relative grading classes to a mix of classes in which some are absolute grading classes and some are relative grading classes. I assume that players first optimize by simultaneously distributing efforts, which corresponds to troops in the original Blotto game, across absolute grading classes and relative grading classes and then optimize by playing a Blotto game in the relative grading classes given that there are at least two relative classes. Seeing the possible payoffs from this scenario, players find the Nash equilibrium of effort distribution across classes.

I find this inquiry relevant to discussions around policies that involve contemplation of replacing an incumbent reward scheme to better distinguish agents on their performance; one of which is the current ongoing discussion in education to completely abolish absolute grading in favor of relative grading, formally known as rank order grading, to combat grade inflation such as Cherry and Ellis (2005).

In fact, if a college department tries to test out relative grading classes to potentially replace absolute grading classes by replacing the grading scheme only for some classes, I believe competition between students may play out as the games I discuss in the paper. Such situations do not seem unlikely given the expensive, complex nature of conducting random trials relevant to the policy in labs.

In particular, if one were to document performance of students across two different classes, they may account for hidden performance bias towards some classes if the game between students dictates that they would be better off if they devoted efforts only towards some classes by considering analysis of the games of this paper.

I also mention additional past literature discussing grading or ranking schemes in regards to instructor bias in a principal agent relation contract. One relatively recent paper Frankel (2014) considers what grading scheme a principal, which corresponds to a university administration that hires instructors to teach and grade for its courses, can mandate to ensure production of the least amount of grading bias, under the assumption that the instructor (agent) for a course has a certain utility function for assigning grades. Alternative forms of the problem of effort allocation across multiple fields in competition against an opponent in the form of bids, where the winner takes all, have been discussed in papers such as Moldovanu and Sela (2006).

1 The Model

1.1 Default Assumptions

I go over common assumptions that apply across all models in the subsequent sections. These will be mentioned again as deemed necessary in subsequent sections.

Assume there are two players, student Blotto and the Enemy student who are taking at least two classes that have grading schemes that is either absolute or relative. Each class (or battlefield) is referred as class n will give a payoff 'a_n' given the effort criterion for the class is satisfied. If player Blotto fulfills the criterion of rewards for class 1 for instance, he will receive a payoff of 'a_1'

The absolute grading class has a threshold amount $T(>0)$ of effort for which a student is able to receive a payoff only if they invest an effort equal or above the effort threshold regardless of the enemy player's effort. Otherwise, their payoff is the negative of the reward.

$$\text{Payoff for Blotto} = \begin{cases} a_n & \text{if } x \geq T \\ -a_n & \text{if } x < T \end{cases}. \tag{1}$$

On the other hand, the relative grading class has a zero sum game payoff where the payoff for Blotto is determined by the amount of effort x and y, given by each student respectively in class n, as such:

$$\text{Payoff for Blotto} = \begin{cases} a_n & \text{if } x > y \\ 0 & \text{if } x = y \\ -a_n & \text{if } x < y \end{cases}. \tag{2}$$

The reward scheme works the same way for the enemy student as well. Payoff for each class may differ to reflect the importance of each classes to the player. For example, the difference in importance could be the difference of an elective and a mandatory class. This is reflected as $a_2 > a_1$ where the elective class is class 2 and the mandatory class is class 1.

Assume both students have a fixed amount of effort that they can distribute between classes. They are referred to as $B(>0)$ and $E(>0)$ for each respective player. I refer to effort given by each player for one of the classes as x and y respectively. For example, when there are two classes, I refer to x and y as the amount of effort given to class 1 and subsequently $B-x$ and $E-y$ as the effort given to class 2.

Payoff for increasing effort is assumed to be constant returns and linear. That is, there is no diminishing mechanism to deduct an effort x to count as a lower effort if it is bigger than a certain amount. There are no differences in efficiency; an effort by Blotto has the same impact as the Enemy. Players also commit all effort they have into the classes. They do not keep some effort unused for leisure.

I will first start with single type classes that are entirely absolute or relative grading classes. Players can observe the other player's effort reserves and simultaneously choose a strategy. Once I reach the mixed classes case however, I assume that the players first decide how much effort they will allocate towards relative classes based on the payoff they can receive by playing a Blotto Game with the enemy player in the relative classes; this will be mentioned in detail in section 1.5. In fact, I can treat it as if the player is deciding to allocate effort between two classes where class 1 is the collection of absolute grading classes and class 2 is the collection of relative grading classes.

1.2 Single Type Classes: Absolute Grading

Consider a case when $n = 2$. Regardless of what the other player does, what the other player's available effort level E is, or what the threshold t is for each class, Blotto can freely choose to allocate efforts between the two classes such that his payoff is maximized. If he cannot reach thresholds for both classes, he will at least give the threshold amount effort for the class with the higher payoff and be indifferent about giving more effort to the higher payoff class. This problem can be easily generalized for n absolute classes as such:

$$T_n(x) = \begin{cases} 1 & \text{if } x \geq t_n \\ 0 & \text{if } x < t_n \end{cases}, \tag{3}$$

$$\max \sum_{i=1}^{n} T_i(x_i)a_i, \quad x_1 + \cdots + x_n = B. \tag{4}$$

Remark. *In the generalized statement of the problem, $T_n(x)$ is an indicator function that is different from T. Think of t_n as T for each absolute grading class.*

1.3 Single Type Classes: Relative Grading

This situation is essentially a two player Blotto game. I cite some parts from Gross and Wagner (1950) to explain the details of the two player Blotto game for the reader and only consider a case when $n = 2$ for the rest of this paper. I refer to effort given to class 1 as x and, subsequently, effort given to class 2 as $B - x$. I also assume the payoff $a_2 \geq a_1$ and let $c = a_2/a_1$.

I first consider a case where the effort reserves of student Blotto and the enemy student are the same and then consider a case in which they are different.

Case $B = E$:

Theorem 1.1. *When $B = E$, student Blotto's optimal strategy is*

$$F^*(x) = I_0(x)$$

and his payoff will be

$$\begin{aligned} K^*(x,y) &= a_1 \operatorname{sgn}(x-y) + a_2 \operatorname{sgn}((B-x)-(E-y)) \\ &= (a_1 - a_2)\operatorname{sgn}(x-y), \end{aligned}$$

Remark. *Here, $I_0(x)$ is a CDF that holds value 1 for random variable x with value 0 and above.*

student Blotto's optimal strategy is to commit all his resources to class 2, hence $B - x = B$ or $x = 0$. For any strategies where Blotto commits any degree less to class 2, he is bound to be dominated by a strategy of the enemy that commits more than Blotto to class 2, which is equivalent to $B - x < E - y$ and $x > y$ leading to a payoff of $a_1 - a_2 \leq 0$ while the enemy student receives $a_2 - a_1 > 0$

The same holds for the enemy student,thus with both committing all effort to class 1 and no effort to class 2, the payoffs for both are K*(x,y) = H*(x,y) = 0.

Case $B > E$: Let the difference between the effort reserves of B and E be denoted $d = B - E$, and let $m \in \mathbb{Z}$ and the remainder r be such that

$$B = md + r \quad (0 \leq r < d).$$

Let $r < p < d$ and let $s = \sum_{j=0}^{m-1} c^j$.

Theorem 1.2. *(Gross and Wagner (1950))*

In the $B > E$ case the payoff for Blotto is

$$v = (a_1/s)(c^m + 1) = \begin{cases} \frac{2a_2}{m} & \text{if } a_2 = a_1 \\ \frac{a_2^m + a_1^m}{a_2^m - a_1^m}(a_2 - a_1) & \text{if } a_2 > a_1 \end{cases}, \tag{5}$$

while the payoff for the Enemy is $-v$.

The optimal strategy for student Blotto is

$$F^*(x) = \frac{1}{s}\sum_{k=1}^{m} c^{m-k} I_{p+(k-1)d}(x)$$

and the optimal strategy for the enemy student is

$$G^*(y) = \frac{1}{s}\sum_{j=0}^{m-1} c^j I_{jq}(y), \quad (q = \frac{E}{m-1}).$$

Remark. *$I_{p+(k-1)d}(x)$ and $I_{jq}(y)$ represent CDFs that have value 1 for random variables with value of effort allocation $p+(k-1)d$ and jq and above respectively and 0 for anything below.*

A more comprehensive proof is available in Gross and Wagner (1950). I go through three main cases, that vary on how much is B exactly bigger than E, to consider to help the reader better understand how these the optimal strategies and payoffs work.

Subcase $B > 2E$:
Corollary 1.2.1. *When $B > 2E$, Blotto's optimal strategy is $F*(x) = I_p(x)$ with a payoff of $a_1 + a_2$. The enemy is indifferent between all strategies and gains a payoff of $-a_1 - a_2$*

Intuitively, this means that student Blotto has effort reserves more than enough to comfortably cover both class 1 and 2. He is indifferent between strategies as long as he gives effort strictly larger than E to each class. Meanwhile the enemy is indifferent between all strategies since he cannot make a difference whatever he does.

This is demonstrated as such.

$$B > 2E$$
$$d = B - E > E$$

Since $0 \le r < d$, it must be that $m = 1$. Otherwise, it would lead to a contraction that $2(B-E) \le B$ which gives us $B \le 2E$. Note that

$$B = d \cdot 1 + r = (B-E)\cdot 1 + r \quad (\Leftrightarrow) \quad r = E$$

Thus, $v = a_1 + a_2$, reflecting Blotto winning in both classes and $F^*(x) = I_p(x)$. See that $F^*(x)$ is 0 if $x < r = E$ to reflect Blotto never allocating less effort to each class than the enemy student's total available effort.

Subcase $B \leq 2E$:

Remark. *As $B \leq 2E$ is a general collection of cases, I go through it breaking it down into cases with distinct optimal strategies and payoffs. Note that they sill follow the structure of theorem 1.2.*

Intuitively, this would mean that student Blotto is no longer able to comfortably dominate the enemy student in both fields. It is not apparent whether simply concentrating all his troops in classroom 2 is a better option than following a mixed strategy where with some probability he commits more to classroom 1 or classroom 2.

I first explore student Blotto's strategy using a payoff matrix. I start with the case $B = 2E$ and let $B = 6$ and $E = 3$ as an example. For the enemy's strategy, I omit any strategies that do not commit completely to either class 1 or class 2 like (1,2) because these are easily dominated by Blotto's strategy like (3,3). Thus, strategies worth exploring would be (3,0) or (0,3). Meanwhile, Blotto has three strategies: commit more to either class (C1 or C2) or split it even (3,3). Note that C1 would be a range of allocations for which there is strictly more effort given to class 1 that student Blotto is indifferent between. For the split even option, which against enemy student's strategies (3,0) and (0,3) results in payoffs a_2 and a_1 respectively, see that they are dominated by Blotto's alternative strategies of committing more to either class 1 or class 2.

		Player E	
		$(3,0)$	$(0,3)$
Player B	$C1$	$(a_1 + a_2, -a_1 - a_2)$	$(a_1 - a_2, -a_1 + a_2)$
	$C2$	$(-a_1 + a_2, a_1 - a_2)$	$(a_1 + a_2, -a_1 - a_2)$

Now I have simplified it down to a form of a penny matching game where both players are now indifferent between their two strategies. In fact, after calculating the optimal mixed strategy using the matrix, find that it is as such

$$F^*(x) = \frac{a_2}{a_1 + a_2} I_p(x) + \frac{a_1}{a_1 + a_2} I_{p+d}(x)$$

where $I_p(x)$ and $I_{p+d}(x)$ correspond to strategy $C2$ and $C1$ respectively and the optimal strategy dictates that the probability either are played should be weighed by the payoffs of winning each class. For our $B = 6$ and $E = 3$ case it would correspond to

$$F^*(x) = \frac{a_2}{a_1 + a_2} I_p(x) + \frac{a_1}{a_1 + a_2} I_{p+3}(x), \quad (0 < p < 3)$$

Meanwhile, for the enemy student the optimal strategy is

$$G^*(y) = \frac{a_1}{a_1+a_2} I_0(y) + \frac{a_2}{a_1+a_2} I_q(y)$$

which for our $B = 6$ and $E = 3$ case would correspond to

$$G^*(y) = \frac{a_1}{a_1+a_2} I_0(y) + \frac{a_2}{a_1+a_2} I_3(y).$$

Once the expected payoffs are calculated, it is equal to (4) which gives us

$$\frac{a_2^2 + a_1^2}{a_2 - a_1}.$$

for Blotto and the negative of Blotto's payoff for the enemy student.
Remark. *Also see that*

$$\frac{a_2^2 + a_1^2}{a_2 - a_1} > \frac{a_2^2 - a_1 a_2}{a_2 - a_1} = a_2,$$

which denies our initial intuition that Blotto committing to class 2 is enough based on the given situation that $a_2 \geq a_1$ and Blotto has more effort reserves.

Our next step is the range $\frac{3E}{2} < B < 2E$.

This step results in the same payoff matrix for as $B = 2E$ case except for the fact that the optimal mixed strategy for Blotto has a different range of p since $r < p < d$ and the remainder would be positive now. Other range of strategies that allocate effort outside the range of p ($r < p < d$) and the range of $p + d$ ($r + d < p + d < 2d$) will be dominated and removed from the set of strategies considered for the mix.

Our next step is $B = \frac{3E}{2}$.

With the same procedure I went through in the two strategies case in the $B = 2E$ case, ruling out for dominated strategies I can find now find a rock scissors paper game where players are indifferent between 3 respective set of strategies. As an example, I choose case $B = 6, E = 4$ for our payoff matrix, which gives me $0 < p < 2$.

		Player E		
		$(0,4)$	$(2,2)$	$(4,0)$
	p	$(a_1 + a_2, -a_1 - a_2)$	$(-a_1 + a_2, a_1 - a_2)$	$(-a_1 + a_2, a_1 - a_2)$
Player B	$p+2$	$(a_1 - a_2, -a_1 + a_2)$	$(a_1 + a_2, -a_1 - a_2)$	$(-a_1 + a_2, a_1 - a_2)$
	$p+4$	$(a_1 - a_2, -a_1 + a_2)$	$(a_1 - a_2, -a_1 + a_2)$	$(a_1 + a_2, -a_1 - a_2)$

In more general terms the payoff matrix would look like the following,

		Player E		
		0	q	$2q$
	p	$(a_1+a_2,-a_1-a_2)$	$(-a_1+a_2,a_1-a_2)$	$(-a_1+a_2,a_1-a_2)$
Player B	$p+E/2$	$(a_1-a_2,-a_1+a_2)$	$(a_1+a_2,-a_1-a_2)$	$(-a_1+a_2,a_1-a_2)$
	$p+E$	$(a_1-a_2,-a_1+a_2)$	$(a_1-a_2,-a_1+a_2)$	$(a_1+a_2,-a_1-a_2)$

from which we can calculate an optimal mixed strategy

$$F^*(x) = \frac{a_2^2}{a_1^2+a_1a_2+a_2^2}I_p(x) + \frac{a_1a_2}{a_1^2+a_1a_2+a_2^2}I_{p+d}(x) + \frac{a_1^2}{a_1^2+a_1a_2+a_2^2}I_{p+2d}(x)$$

for Blotto and optimal mixed strategy

$$G^*(y) = \frac{a_1^2}{a_1^2+a_1a_2+a_2^2}I_0(x) + \frac{a_1a_2}{a_1^2+a_1a_2+a_2^2}I_q(y) + \frac{a_2^2}{a_1^2+a_1a_2+a_2^2}I_{2q}(y) \quad (q = \frac{E}{3-1})$$

for the enemy student.

The next case would be $\frac{4E}{3} < B < \frac{3E}{2}$ which has the same optimal mixed strategy with a different p in a slightly different range with the remainder being positive again. Then, consider case $B = \frac{4}{3}$ in which Blotto will have 4 strategies for which he is indifferent between, will mix weighting them by a probability according to the payoff of each class in a similar fashion. Then, continue on to $\frac{5}{4}$, $\frac{6}{5}$, $\frac{7}{6}$ and so on while never reaching $B = E$ since we have assumed $B > E$. With the same procedure, see that there are increasing number of strategies for Blotto and the enemy for which they are indifferent.

1.4 Mixed Classes: $n = 2$, Equal Thresholds

Now I have student Blotto and the enemy student taking a mix of classes. The first case I consider is a mix of two classes where one is an absolute grading class while the other is a relative grading class.

Remark. *This case does not involve a Blotto game and the same holds for section 2.3, which will have different threshold Ts. This is due to the fact that there is only 1 relative class, which does not call for another distribution strategy on top of the distribution between the absolute grading classes and the relative grading classes.*

For the rest of the mixed classes case, I maintain the assumption that $B = E$. As mentioned before, I refer to x and y as the amount of effort given to class 1 by Blotto and the enemy student respectively and subsequently $B - x$ and $E - y$ as the effort given to class 2 by Blotto and the enemy student respectively. Let class 1 be a relative grading class and let class 2 be an absolute grading class.

Let the absolute threshold T for the absolute class be equal for both players. There are three strategies for student Blotto to consider: $B - x > T$, $B - x = T$, $B - x < T$. Since investing more effort beyond the threshold value for the absolute class yields no more payoff than simply investing effort at the threshold level T and takes away effort from the relative class, Blotto will not choose $B - x > T$. The same holds for the enemy student. Then I am left with a payoff matrix as such.

		Player E	
		$E - y = T$	$E - y < T$
Player B	$B - x = T$	(a_2, a_2)	$(-a_1 + a_2, a_1 - a_2)$
	$B - x < T$	$(a_1 - a_2, -a_1 + a_2)$	$(-a_2, -a_2)$

Remark. *I clarify some details that are missing from the payoff matrix. It should be noted that the $B - x < T$ and $E - y < T$ actually encompass a range of strategies for which players invest effort below T to the absolute grading class 2. If both players choose to put effort below T in the absolute grading class, however, neither player has the incentive to simply put only some effort into the relative classes as it is easily dominated by the other player putting more effort into the relative grading class, which results in a negative payoff for the player with the lower effort.*

Thus, once either player chooses the strategy to invest less than T effort into the absolute grading class, they have incentive to invest all effort into the relative grading class. If players both decide not to meet the threshold, they will both invest all effort into the relative class, resulting in equal efforts in the relative class. This ultimately results in both receiving a payoff of $-a_2$.

Proposition. *There are 2 different pure strategy equilibria $B - x = T$ vs $E - y = T$ and $B - x < T$ vs. $E - y < T$ dependent on the payoffs of class 1 and 2.*

Case $a_2 > a_1$: both players have incentive to play the threshold and each receive payoff a_2 as the effort they put in the relative grading classes are the same which results in the **case** $B = E$ mentioned in section 1.3.

Case $a_1 > a_2$: For the first case $a_1 - a_2 > a_2$, which implies $a_1 > 2a_2$, both players put all effort into the relative grading class if payoff a_1 is sufficiently big enough. Otherwise, if $a_1 - a_2 < a_2$ they both still play the threshold.

Case $a_1 = a_2$: Trivially, both players play the threshold.

The results in section 1.4 seem to align well with intuition. If the absolute grading class has higher payoffs, there is no reason to deviate from playing the threshold just to risk penalty far greater than the maximum reward from the relative class. If the relative grading class has higher payoffs and has payoff sufficiently high enough to negate the impediments of payoffs being dependent on the opponent's chosen strategy, players put all effort in the relative grading class. Unfortunately, both players are worse off relative to the best possible equilibrium without benefiting from putting all their effort into the relative class.

1.5 Mixed Classes: n=3, Two Relative Classes, Equal Thresholds

For this case, there 3 classes two of which are relative grading classes. Class 1 and 2 are relative grading classes and class 3 is an absolute grading class. I assume $B = E$ and the threshold Ts for both students are equal.

As mentioned in section 1.1, I assume that students first decide to allocate how much effort they will invest to the relative classes in total based on optimizing with respect to the payoffs they expect to receive from the relative classes by playing a Blotto game with the enemy student. I now call this total effort given to relative class 1 and 2 as x and subsequently the rest of effort given to the absolute class 3 is $B - x$.

Players can observe the enemy player's effort reserves and both players simultaneously pick the strategy of whether to play the threshold or put all effort into the relative classes (which I will explain why in the following remark). Next, they observe how much effort the enemy has allocated towards the relative classes. Both players simultaneously pick a strategy of effort allocation across the two relative classes, choosing an optimizing strategy and receiving a payoff according to theorem 1.2 in section 1.3 dependent on whether $x > 2y$, $x = 2y$, $y < x < 2y$ or vice versa respectively. These correspond to **case** $B > 2E$, $B = 2E$, and $E < B < 2E$ in section 1.3.

The payoff matrix is as such.

		Player E	
		$E-y=T$	$E-y<T$
Player B	$B-x=T$	(a_3, a_3)	$(-a+a_3, a-a_3)$
	$B-x<T$	$(a-a_3, -a+a_3)$	$(-a_3, -a_3)$

Remark. *Again, I describe missing details in the matrix. It is different from section 1.4. It should be noted that the $B-x<T$ and $E-y<T$ actually encompass a range of strategies for which players invest effort below T for the absolute grading class 3. But neither player has the incentive to simply put some effort into the relative classes as we have seen in the classical Blotto game's $n=2$, $B>E$ case, having less available effort for the relative grading classes compared to the competing player results in strictly less payoffs compared to when they have the same amount of effort available for the relative grading class and even less when the difference in effort is bigger.*

*Thus once a player makes the decision to invest less than T effort into the absolute grading class, they have incentive to invest all effort into the relative grading class. In particular, both players earn a payoff of $-a_3$ for failing to meet threshold T and a payoff 0 from engaging in the Blotto game in the relative classes 1 and 2 in a **case** $B=E$ as mentioned in section 1.3, thus $-a_3$ total payoff as seen in the payoff matrix.*

There are different pure strategy equilibria depending on the payoffs. It may in fact look similar in form to the $n=2$ case in section 1.4. Recall, however, that I have assumed that players first decide to distribute efforts between the absolute grading class and the relative grading class after considering the payoffs they will earn from playing the Blotto game in the two relative grading classes for some effort from the reserves put up against expected effort investments from the enemy player. In other words, the payoffs are now not a simple matter of a_1 or $-a_1$ as in section 1.4 based on who has more effort in classroom 1.

I will, thus, also have to consider threshold T to be a deciding factor of what a is for the asymmetric strategy cases in the payoff matrix. This is because based on the magnitude of T, the decisions $B-x<T$ against $E-y=T$ and $E-y<T$ against $B-x=T$ yields different x vs. y situations that may range from $B>2E$ to $E<B\leq 2E$ cases mentioned in section 1.3.

Specifically, there are two cases:

Case $T>\frac{1}{2}B=\frac{1}{2}E$: The interaction of strategy $B-x<T$ against $E-y=T$

yields $a = a_1 + a_2$ as $E - y = T$ now implies $y = E - T < \frac{1}{2}E = \frac{1}{2}B$ which is equivalent to $2y < B$. This is the case $B > 2E$ section 1.3 where I mentioned Blotto comfortably dominating both classes and getting a payoff $a_1 + a_2$ across both classes. The same applies for interaction of strategy $B - x = T$ against $E - y < T$ case as T is equal for both players.

In fact, see that specific for this case only, the equilibria is similar to section 1.4.
Proposition. *There are two different pure strategy equilibrium dependent on the relationship between payoffs of class 3 and total payoff $a_1 + a_2$ from the relative classes in the case $T > \frac{1}{2}B = \frac{1}{2}E$.*

Subcase $a_3 > a = a_1 + a_2$: both players have incentive to play the threshold and each receive payoff a_2 as the effort they put in the relative grading classes are the same which results in the $B = E$ situation mentioned in section 1.3.

Subcase $a = a_1 + a_2 > a_3$: Since $a_1 + a_2 - a_3 > a_3$ implies $a_1 + a_2 > 2a_3$, both players put all effort into the relative grading class if payoff $a_1 + a_2$ is sufficiently big enough. Otherwise, they both still play the threshold.

Subcase $a = a_1 + a_2 > a_3$: Trivially, both players play the threshold.

Case $T \leq \frac{1}{2}B = \frac{1}{2}E$: Now, by the same logic in the $T > \frac{1}{2}B = \frac{1}{2}E$ case, see that in the asymmetric strategies cases, $B \leq 2E$ cases mentioned in section 1.3 arise and thus the payoffs a for the player with more effort in the relative classes $-a$ for the other player, following equation (5) in theorem 1.2, are something smaller than $a_1 + a_2$.

Players find equilibria in a similar fashion but now the relation between a and a_3 must be reconsidered for each case depending on specifically how big threshold T is as a varies with threshold T in interaction of strategies $B - x < T$ against $E - y = T$ and interaction of strategies $B - x = T$ against $E - y < T$. I at least know that $0 < a < a_1 + a_2$

Subcase $a_3 < a - a_3$: This also implies $-a_3 > -a + a_3$ Both players put all effort into the relative classes.

Subcase $a_3 > a - a_3$: This also implies $-a_3 < a - a_3$. Both players play the threshold.

Subcase $a_3 = a - a_3$: This also implies $-a_3 = a - a_3$. Players are indifferent between their available strategies.

2 Model Modification: Different Thresholds

So far, I have considered cases where threshold T was equal for both players. I now consider cases where the threshold for the absolute class is different for each student. In terms of the real world, it may be seen as partially incorporating efficiency of effort. I set it up such that Blotto must put in more effort to achieve good grades compared to the enemy student in the absolute grading class.

I denote these thresholds T^B and T^E for Blotto and the enemy student respectively. I assume $B = E$ and inherit the respective x and $B - x$ notation for cases $n = 2$ and $n = 3$ from sections 1.4 and 1.5 respectively.

2.1 Mixed Classes: $n = 2$, $T^B > T^E$

Assuming Blotto's threshold is higher than the enemy's threshold for class 2 with other assumptions remaining the same, I get the following payoff matrix.

		Player E	
		$E - y = T^E$	$E - y < T^E$
Player B	$B - x = T^B$	$(-a_1 + a_2, a_1 + a_2)$	$(-a_1 + a_2, a_1 - a_2)$
	$B - x < T^B$	$(a_1 - a_2, -a_1 + a_2)$	$(-a_2, -a_2)$

Due to the introduction of the difference in threshold, there are more ambiguities in determining which payoff is bigger on top of different total payoffs depending on the payoff each class that must be addressed. I start with the more simple case.

Proposition. *In the mixed two classes case with student Blotto having a higher threshold, there are three different pure strategy equilibria $B-x = T^B vs. E-y = T^E$, $B - x < T^B vs. E - y = T^E$, and $B - x < T^B vs. E - y < T^E$ depending on the payoffs of class 1 and 2.*

Case $a_2 > a_1$: It is clear that both players playing the threshold is the equilibrium.

Case $a_1 > a_2$: There are two cases. If $-a_1 + a_2 > -a_2$, combined with $a_2 < a_1$, it implies $a_2 < a_1 < 2a_2$. The equilibrium is for Blotto to put all effort into the relative grading class ($B - x < T^B$) and for the enemy to put effort into the absolute grading class. ($E - y = T^E$)

On the other hand, if $-a_2 > -a_1 + a_2$, which implies $2a_2 < a_1$, the equilibrium is both players putting all effort into the relative grading class. ($B - x < T^B$ vs. $E - y < T^E$)

Case $a_1 = a_2$: Trivially, both players play the threshold.

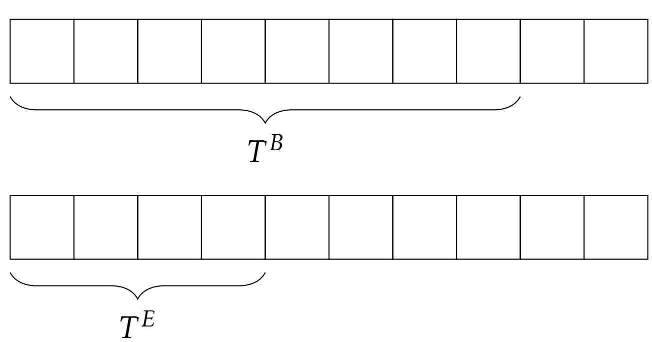
With difference in absolute threshold, intuitively, it might seem that student Blotto is at a disadvantage. Playing the threshold automatically forces him to allocate less effort the relative classes and, on the other hand, if the relative classes payoffs are high, both players will choose to invest all effort into the relative classes and both be worse off. There seems to be no outcome where Blotto is better off.

In fact, however, despite facing a disadvantage solely from competing in the absolute threshold grading class by having a higher threshold, if the rewards are set just right such that $a_2 < a_1 < 2a_2$, there is an equilibrium where student Blotto is better off than the enemy student with a payoff of $a_1 - a_2 > 0$ while the enemy student earns a payoff of $-a_1 + a_2 < 0$.

2.2 Mixed Classes: $n = 3$, $T^B > T^E$

I now consider a modification of section 1.5. On top of the different thresholds T^B and T^E, I must also consider how exactly big they are and how they relate to each other.

There are three cases to consider $E - T^E > 2(B - T^B)$, $E - T^E = 2(B - T^B)$, and $E - T^E < 2(B - T^B)$. In the following figure, I present a visualization of one specific case of $E - T^E > 2(B - T^B)$ to help visualize how the thresholds affect a specific equilibrium I next mention.



For latter two cases, the number ambiguities increase as not only do I have to consider the difference in threshold values and relationship between $a_1 + a_2$ and a_3 I must also consider cases where the payoff is between 0 and $a_1 + a_2$ as I will demonstrate for the first case. This makes this problem hard and lengthy to state a definitive equilibrium by approaching it by each case. Instead, I first present a general optimal strategy and equilibrium that holds across all cases. Next, I will go over a specific case $E - T^E > 2(B - T^B)$ showing a definitive case in which there exists an interesting equilibrium under the right conditions.

Proposition. *There exist three different pure strategy equilibriums when the payoffs $v - a_3$ and $a_3 - (a_1 + a_2)$ are strictly different (i.e. either $<$ or $>$) in the mixed three classes case with two relative grading classes and different thresholds.*

The payoff matrix is as such.

		Player E	
		$E - y = T^E$	$E - y < T^E$
Player B	$B - x = T^B$	$(-(a_1 + a_2) + a_3, a_1 + a_2 + a_3)$	$(-(a_1 + a_2) + a_3, a_1 + a_2 - a_3)$
	$B - x < T^B$	$(v - a_3, -v + a_3)$	$(-a_3, -a_3)$

Remark. *Note that v follows the payoff equation (5) in theorem 1.2 depending on the relationship between B and $E - T^E$, the efforts put into the relative classes by each respective player in the interaction of strategies $B - x < T^B$ vs. $E - y = T^E$, which corresponds to the effort reserves B and E in section 1.3's Blotto game. Thus* $0 < v \le a_1 + a_2$

Case $a_1 + a_2 < 2(a_1 + a_2) < a_3$ and $a_1 + a_2 < a_3 < 2(a_1 + a_2)$: Both players will play the threshold as regardless of what v is, $v - a_3$ is sure to be less than 0. In this equilibrium the absolute class's payoff forces Blotto to allocate less effort towards the relative classes to such a degree a **case** $B > 2E$ from section 1.3 occurs and the enemy player is better off.

Case $a_3 < a_1 + a_2 < 2(a_1 + a_2)$:

Subcase $v - a_3 < a_3 - (a_1 + a_2)$: This implies $-a_3 < a_3 - (a_1 + a_2)$. This also implies $-v + a_3 > -a_3 + (a_1 + a_2)$ which in turn implies $a_3 - v > -a_3$. Thus, both players play the threshold.

Subcase $v - a_3 > a_3 - (a_1 + a_2)$: If $-a_3 > a_3 - (a_1 + a_2)$, both players put all effort into the relative classes. If $-a_3 < a_3 - (a_1 + a_2)$, the equilibrium is $B - x < T^B$ vs. $E - y = T^E$. If $-a_3 = a_3 - (a_1 + a_2)$, which would also imply $-v + a_3 \ge -a_3 =$

$a_3 - (a_1 + a_2)$. In subsubcase $-v + a_3 > -a_3 = a_3 - (a_1 + a_2)$, the equilibrium is $B - x < T^B$ vs. $E - y = T^E$. In subsubcase $-v + a_3 = -a_3 = a_3 - (a_1 + a_2)$, the enemy player is only indifferent between his strategies if Blotto chooses strategy $B - x < T^B$ while student Blotto is only indifferent between his strategies if the enemy chooses strategy $E - y < T^E$. While Blotto is indifferent between his strategies, if he chooses to play the threshold when the enemy student chooses not to play the threshold, the enemy player will be worse off. Thus the enemy player chooses to play the threshold. Seeing this, in turn, Blotto will be incentivized to choose $B - x < T^B$. The equilibrium is thus again $B - x < T^B$ vs. $E - y = T^E$.

Subcase $v - a_3 = a_3 - (a_1 + a_2)$: This implies $-a_3 < a_3 - (a_1 + a_2)$. This also implies $a_3 - v = -a_3 + (a_1 + a_2)$ which in turn implies $a_3 - v > -a_3$. Student Blotto is indifferent between his strategies only when the Enemy student chooses to play the threshold $E - y = T^E$. The enemy player on the other hand, has his all effort into the relative classes dominated by choosing the threshold and thus chooses to play $E - y = T^E$.

Now I go over the specific case of $E - T^E > 2(B - T^B)$.

Case $E - T^E > 2(B - T^B)$: The payoff matrix is as such.

		Player E	
		$E - y = T^E$	$E - y < T^E$
Player B	$B - x = T^B$	$(-(a_1 + a_2) + a_3, a_1 + a_2 + a_3)$	$(-(a_1 + a_2) + a_3, a_1 + a_2 - a_3)$
	$B - x < T^B$	$(a - a_3, -a + a_3)$	$(-a_3, -a_3)$

Proposition. *In case $E - T^E > 2(B - T^B)$, there exists a equilibrium where student Blotto is better off than the enemy student. Thus in general for the mixed three classes with two relative classes and different absolute grading classes thresholds in which student Blotto has a higher threshold T^B, there exists a equilibrium where student Blotto is better off than the enemy student.*

Note that a in the payoff matrix now is $0 < a \leq a_1 + a_2$. This is due to fact difference in threshold of the two players adds ambiguity in how effort B, given student Blotto has incentive to put all effort into the relative class and chooses strategy $B - x < T^B$, is bigger compared to effort y that the enemy student puts into classes 1 and 2. It could be that $B > 2y$ or not. It is at least certain that $B > y$, thus by the results we saw in section 1.3—specifically the **cases** $B \leq 2E$ and $B = 2E$—we can make a relatively weak assertion that $0 < a \leq a_1 + a_2$.

For subcases $a_1 + a_2 < 2(a_1 + a_2) < a_3$ and $a_1 + a_2 < a_3 < 2(a_1 + a_2)$, both players will play the threshold as regardless of what a is, $a - a_3$ is sure to be less than 0. In this equilibrium the absolute class's payoff forces Blotto to allocate less effort towards the relative classes to such a degree a **case** $B > 2E$ from section 1.3 occurs and the enemy player is better off.

For subcase $a_3 < a_1 + a_2 < 2(a_1 + a_2)$ however, we must consider a's magnitude. If $B > 2y$ for the case $B - x < T^B$ against $E - y = T^E$ and on top of that, if $-a_3 < a_3 - (a_1 + a_2)$ which implies $2a_3 > a_1 + a_2$, see that $B - x < T^B$ against $E - y = T^E$ is in fact the equilibrium.

Thus, I have shown that if the payoffs are set just right such that $a_3 < a_1 + a_2 < 2a_3 < 2(a_1 + a_2)$ and the threshold relation is $E - T^E > 2(B - T^B)$, where Blotto has a very large threshold value, we see an equilibrium where student Blotto, who seems to be at an disadvantage to the enemy student, is better off compared to the enemy student with a payoff of $a - a_3 = a_1 + a_2 - a_3 > 0$ while the enemy student receives a payoff of $a - a_3 = a_1 + a_2 - a_3 < 0$

3 Conclusion

I have explored the optimal allocations of student effort across courses with different grading schemes under the assumption of constant returns to effort and players first deciding to split effort between the absolute grading class and the relative grading class based on expected payoffs from playing the Colonel Blotto game in the relative grading classes.

There are certainly limitations to this paper. Constant returns to effort fails to reflect decreasing marginal returns with more effort. It also fails to reflect the different efficiency with which students can transfer effort into payoffs. Adding such a feature would potentially ruin the zero sum game framework and complicate calculations. Adding more absolute classes and relative classes is certainly another realm I could have explored.

I was able to address the basic dynamics of student effort allocation should situations in which students take two classes with different grading schemes arise. I have also demonstrated the possibility of an existence of interesting equilibrium, where the seemingly disadvantaged student is better off, under the condition that we incorporate the fact that students may have to put in more effort to reach the same grade in the absolute grading class and we set the payoffs of the classes just right.

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